Two-person random race

Alice and Bob play the following game: they take turns to collect 1 or 2 chips randomly and independently with equal probability, with Alice going first. The first person who collects at least n chips is the winner. We have determined that the winning probability of Bob is

$$\frac{1}{2} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{4^k}\left(\binom{k}{n-k} + \binom{k-1}{n-k}\right)^2. \tag{1}$$

We have also generalized the problem. If Alice and Bob take turns to collect a_1, a_2, $\ldots$, or a_m chips randomly and independently with probabilities $p_1, p_2, \ldots, p_m$ respectively, where $a_1 < a_2 < \cdots < a_m$ are positive integers, $p_1, p_2, \ldots, p_m \in [0,1]$, and $p_1 + p_2 + \cdots + p_m = 1$, then the winning probability of Bob is

$$\frac{1}{2} - \frac{1}{2} \cdot \sum_{k=1}^{\left\lceil \frac{n}{a_1} \right\rceil} \left(\sum_{\substack{(s_{k1}, s_{k2}, \ldots, s_{km}) \in \mathbb{Z}_{\geq 0}^m: \\ \sum_{g=1}^m s_{kg} a_g = n \\ \sum_{g=1}^m s_{kg} = k}} \binom{k}{s_{k1}, s_{k2}, \ldots, s_{km}} \prod_{\ell=1}^{m} p_\ell^{s_{k\ell}} \right.$$

$$\left. + \sum_{j=1}^{m} \sum_{i=a_{j-1}}^{a_j - 1} \sum_{\substack{(s_{k1}, s_{k2}, \ldots, s_{km}) \in \mathbb{Z}_{\geq 0}^m: \\ \sum_{g=1}^m s_{kg} a_g = n - i \\ \sum_{g=1}^m s_{kg} = k-1}} \binom{k-1}{s_{k1}, s_{k2}, \ldots, s_{km}} \prod_{\ell=1}^{m} p_\ell^{s_{k\ell}} \left(\sum_{\alpha=j}^{m} p_\alpha \right) \right)^2.$$

There are a number of future directions in this project.

1. Find the closed form of Equation (1)
2. Find the winning probability of Bob if the winner must collect exactly n chips (if a player collects more than n chips, then they either "bounce back" like the game of Sorry, or they loop back to the beginning like modular arithmetic).
3. Find the winning probability of Bob if the number of chips collected can be 0 or negative.
4. Introduce more players.

References

[1] T. W. H. Wong and J. Xu, A probabilistic take-away game, *J. Integer Seq.* **21** (2018), Article 18.6.3.